Diff. eqn, B.Sc. Part III

7th paper

$\boxed{\text{LDE}}$

Q. Solve $x\dfrac{d^2y}{dx^2}+(1-x)\dfrac{dy}{dx}-y=e^x$.

Soln The given equation

$$\frac{d^2y}{dx^2}+\frac{1-x}{x}\frac{dy}{dx}-\frac{y}{x}=\frac{1}{x}e^x$$

which is of the form

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

Here, $P=\dfrac{1-x}{x}$, $Q=\dfrac{-1}{x}$, $R=\dfrac{1}{x}e^x$

Now, $1+P+Q=1+\dfrac{1-x}{x}-\dfrac{1}{x}=\dfrac{\cancel{x}+\cancel{1}-\cancel{x}-\cancel{1}}{x}=0$

$\Rightarrow y=e^x$ is a part of CF.

Let $y=uv$ is the soln of given eqn, where $u=e^x$.

$\therefore y=e^x.v$

Now, v is given by

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left(\frac{1-x}{x}+2.e^{-x}.e^x\right)\frac{dv}{dx}=\frac{1}{x}e^x.\frac{1}{e^x}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{1-x}{x} + 2\right)\frac{dv}{dx} = \frac{1}{x}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{1+x}{x}\right)\frac{dv}{dx} = \frac{1}{x}$$

put $\frac{dv}{dx} = z$

$$\Rightarrow \frac{dz}{dx} + \left(\frac{1+x}{x}\right)z = \frac{1}{x}$$ which is a linear eqn. Its

$$IF = e^{\int \frac{1+x}{x}dx} = e^{\int (\frac{1}{x}+1)dx} = e^{\log x + x} = xe^{x}.$$

Its soln is

$$z \cdot IF = \int \frac{1}{x} \cdot IF \, dx$$

$$\Rightarrow z \cdot xe^{x} = \int \frac{1}{x} \cdot xe^{x} dx \Rightarrow z x e^{x} = e^{x} + K$$

$$\Rightarrow z = \frac{1}{x} + \frac{K}{x}e^{-x} \Rightarrow \frac{dv}{dx} = \frac{1}{x} + \frac{K}{x}e^{-x}$$

$$\Rightarrow dv = \frac{1}{x}dx + \frac{K}{x}e^{-x}dx$$

integrating we get

$$\Rightarrow V = \log x + K\int \frac{1}{x}e^{-x}dx + K_1 \quad \text{—— (A)}$$

where K and K_1 are constants of integrations.

Hence, $y = uv$ where $u = e^{x}$ and v is given by eq (A) is the required soln.